Proofs of Theorems in Exact Mixed-Integer Programming Approach for Chance-Constrained Multi-Area Reserve Sizing

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1 Introduction

In this paper, we show all the proofs of the theorems in the reference paper Exact Mixed-Integer Programming Approach for Chance-Constrained Multi-Area Reserve Sizing. Please notice that the title numbers of equations, definitions, lemmas, and theorems are different from the reference paper. This is an inevitable choice to make this paper as a whole. The Claim 2.1.2 in the reference paper is identical to the Claim 2.5.2 in this paper. The Theorem 2.3 in the reference paper is equivalent to the Theorem 2.6 in this paper. Apart from these two proofs that are missing in the reference paper, we have included all the definitions, lemmas, and theorems in this paper for the reference paper. Notice that Definition 2.3, Lemma 2.3, and Lemma 2.4 are new in this paper, and they are used to prove the Claim 2.5.2.

2 Proofs

Definition 2.1 (Connected Vertex Set)**.** For a graph $\mathcal{G}(V,E)$, the Connected Vertex Set $\mathcal{W}(\mathcal{G})$ is defined as follows:

$$\mathcal{W}(\mathcal{G}) = \{S \subseteq V : \forall v, w \in S, \exists \text{ a path } P \text{ on } \mathcal{G} \text{ s.t. } v, w \in V(P) \subseteq S\}, \tag{1}$$

where $V(P)$ denotes the set of vertices in the path P.

Definition 2.2 (Maximum Input/Output Flow)**.** For a directed graph $\mathcal{G}(V,E)$ where $\forall e \in E, f(e)$ denotes the flow in e and $-T_e^- \leq f(e) \leq T_e^+$, for all $S \subseteq V, E' \subseteq E$, the Maximum Input Flow $I(S)$ and the Maximum Output Flow $O(S)$ on E' are defined as follows:

$$I(S|E') = \sum_{v\in S, w\in S^c:(v,w)\in E'} T^-_{(v,w)} + \sum_{v\in S, w\in S^c:(w,v)\in E'} T^+_{(w,v)}, \tag{2}$$

$$O(S|E') = \sum_{v\in S, w\in S^c:(v,w)\in E'} T^+_{(v,w)} + \sum_{v\in S, w\in S^c:(w,v)\in E'} T^-_{(w,v)}. \tag{3}$$

Lemma 2.1. For a graph $\mathcal{G}(V,E)$,

$$I(S_1 \setminus S_2|E') + O(S_2 \setminus S_1|E') \leq I(S_1|E') + O(S_2|E'), \quad \forall S_1, S_2 \subseteq V, E' \subseteq E.$$

Proof. For the compactness of the proof, without loss of generality, we leave out the conditions $(v,w) \in E'$ or $(w,v) \in E'$ under the summation sign. We can divide $I(S_1 \setminus S_2|E')$ into two terms:

$$I(S_1 \setminus S_2|E') = \sum_{v\in S_1\setminus S_2, w\in S_1^c} (T^-_{(v,w)} + T^+_{(w,v)}) + \sum_{v\in S_1\setminus S_2, w\in S_1\cap S_2} (T^-_{(v,w)} + T^+_{(w,v)}). \tag{4}$$

Observe that since $S_1 \setminus S_2 \subseteq S_2^c$, the second term

$$\sum_{v\in S_1\setminus S_2, w\in S_1\cap S_2} (T^-_{(v,w)} + T^+_{(w,v)}) \leq \sum_{v\in S_2^c, w\in S_1\cap S_2} (T^-_{(v,w)} + T^+_{(w,v)}). \tag{5}$$

By changing v and w, we can obtain

$$\sum_{v\in S_2^c, w\in S_1\cap S_2} (T^-_{(v,w)} + T^+_{(w,v)}) = \sum_{v\in S_1\cap S_2, w\in S_2^c} (T^+_{(v,w)} + T^-_{(w,v)}). \tag{6}$$

In a similar way, $O(S_2 \setminus S_1|E')$ can be divided into two terms:

$$O(S_2 \setminus S_1|E') = \sum_{v\in S_2\setminus S_1, w\in S_2^c}(T^+_{(v,w)} + T^-_{(w,v)}) + \sum_{v\in S_2\setminus S_1, w\in S_1\cap S_2}(T^+_{(v,w)} + T^-_{(w,v)}). \tag{7}$$

Since $S_2 \setminus S_1 \subseteq S_1^c$, the second term

$$\sum_{v\in S_2\setminus S_1, w\in S_1\cap S_2}(T^+_{(v,w)} + T^-_{(w,v)}) \leq \sum_{v\in S_1^c, w\in S_1\cap S_2}(T^+_{(v,w)} + T^-_{(w,v)}). \tag{8}$$

By changing v and w, we can obtain

$$\sum_{v\in S_1^c, w\in S_1\cap S_2}(T^+_{(v,w)} + T^-_{(w,v)}) = \sum_{v\in S_1\cap S_2, w\in S_1^c}(T^-_{(v,w)} + T^+_{(w,v)}). \tag{9}$$

Now observe that the sum of the first term of (4) and the right-hand-side of (9) is equal to $I(S_1|E')$. Likewise, the sum of the first term of (7) and the right-hand-side of (6) is equal to $O(S_2|E')$. Thus, $I(S_1 \setminus S_2|E') + O(S_2 \setminus S_1|E') \leq I(S_1|E') + O(S_2|E')$. □

Lemma 2.2. For a graph $\mathcal{G}(V, E)$ for all $S_1, S_2 \subseteq V, E' \subseteq E$,

$$O(S_1 \cup S_2|E') + O(S_1 \cap S_2|E') = O(S_1|E') + O(S_2|E') - \Phi(S_1, S_2|E')$$

$$I(S_1 \cup S_2|E') + I(S_1 \cap S_2|E') = I(S_1|E') + I(S_2|E') - \Phi(S_1, S_2|E')$$

where

$$\Phi(S_1, S_2|E') = \sum_{v,w\in(S_1\setminus S_2)\cup(S_2\setminus S_1):(v,w)\in E'}(T^+_{(v,w)} + T^-_{(v,w)}).$$

Proof. Since it is almost same, we only show the case of Maximum Output Flow. For the compactness of the proof, without loss of generality, we leave out the conditions $(v, w) \in E'$ or $(w, v) \in E'$ under the summation sign. Notice that $O(S|E')$ is consist of the terms related to $T^+_{(v,w)}$ and those of $T^-_{(v,w)}$. In this proof, the patterns for $T^+_{(v,w)}$ and $T^-_{(v,w)}$ are exactly same and what is important is the relationship of summations, so we omit $T^+_{(v,w)}$ and $T^-_{(v,w)}$ on the course of equations. Notice that the right-hand-side can be written as follows:

$$O(S_1|E')+O(S_2|E')-\Phi(S_1, S_2|E') = \sum_{v\in S_1, w\in S_1^c} + \sum_{v\in S_2, w\in S_2^c} - \sum_{v\in S_1\setminus S_2, w\in S_2\setminus S_1} - \sum_{v\in S_2\setminus S_1, w\in S_1\setminus S_2} \tag{10}$$

Since

$$\sum_{v\in S_1, w\in S_1^c} = \sum_{v\in S_1\setminus S_2, w\in S_2\setminus S_1} + \sum_{v\in S_1\setminus S_2, w\in(S_1\cup S_2)^c} + \sum_{v\in S_1\cap S_2, w\in S_2\setminus S_1} + \sum_{v\in S_1\cap S_2, w\in(S_1\cup S_2)^c} \tag{11}$$

$$\sum_{v\in S_2, w\in S_2^c} = \sum_{v\in S_2\setminus S_1, w\in S_1\setminus S_2} + \sum_{v\in S_2\setminus S_1, w\in(S_1\cup S_2)^c} + \sum_{v\in S_1\cap S_2, w\in S_1\setminus S_2} + \sum_{v\in S_1\cap S_2, w\in(S_1\cup S_2)^c} \tag{12}$$

the first terms of (11) and (12) are crossed out with the third and the fourth term of (10). From the rest of the terms observe that

$$\sum_{v\in S_1\setminus S_2, w\in(S_1\cup S_2)^c} + \sum_{v\in S_2\setminus S_1, w\in(S_1\cup S_2)^c} + \sum_{v\in S_1\cap S_2, w\in(S_1\cup S_2)^c} = \sum_{v\in(S_1\cup S_2), w\in(S_1\cup S_2)^c} \tag{13}$$

$$\sum_{v\in S_1\cap S_2, w\in S_2\setminus S_1} + \sum_{v\in S_1\cap S_2, w\in S_1\setminus S_2} + \sum_{v\in S_1\cap S_2, w\in(S_1\cup S_2)^c} = \sum_{v\in(S_1\cap S_2), w\in(S_1\cap S_2)^c}. \tag{14}$$

The right-hand-side of (13) is $O(S_1 \cup S_2|E')$ and the right-hand-side of (14) is $O(S_1 \cap S_2|E')$. Thus, $O(S_1 \cup S_2|E') + O(S_1 \cap S_2|E') = O(S_1|E') + O(S_2|E') - \Phi(S_1, S_2|E')$. □

Definition 2.3 (Net Output Flow)**.** For a directed graph $\mathcal{G}(V, E)$ where $\forall e \in E, \hat{f}(e)$ denotes the flow in e, for all $S \subseteq V, E' \subseteq E$, the Net Output Flow on E', $\Gamma(S|E')$ is defined as follows:

$$\Gamma(S|E') = \sum_{(v,w)\in E':v\in S}\hat{f}_{(v,w)} - \sum_{(v,w)\in E':w\in S}\hat{f}_{(v,w)}. \tag{15}$$

Lemma 2.3. For a graph $\mathcal{G}(V, E)$,

$$\Gamma(S_1|E') - \Gamma(S_2|E') = \Gamma(S_1 \setminus S_2|E') - \Gamma(S_2 \setminus S_1|E'), \quad \forall S_1, S_2 \subseteq V, E' \subseteq E.$$

Proof. For the compactness of the proof, without loss of generality, we leave out the conditions $(v, w) \in E'$ under the summation sign.

$$\Gamma(S_1|E') = \sum_{v \in S_1 \setminus S_2} \hat{f}_{(v,w)} + \sum_{v \in S_1 \cap S_2} \hat{f}_{(v,w)} - \sum_{w \in S_1 \setminus S_2} \hat{f}_{(v,w)} - \sum_{w \in S_1 \cap S_2} \hat{f}_{(v,w)} \tag{16}$$

$$\Gamma(S_2|E') = \sum_{v \in S_2 \setminus S_1} \hat{f}_{(v,w)} + \sum_{v \in S_1 \cap S_2} \hat{f}_{(v,w)} - \sum_{w \in S_2 \setminus S_1} \hat{f}_{(v,w)} - \sum_{w \in S_1 \cap S_2} \hat{f}_{(v,w)} \tag{17}$$

Observe that

$$\begin{aligned} \Gamma(S_1|E') - \Gamma(S_2|E') &= (\sum_{v \in S_1 \setminus S_2} \hat{f}_{(v,w)} - \sum_{w \in S_1 \setminus S_2} \hat{f}_{(v,w)}) - (\sum_{v \in S_2 \setminus S_1} \hat{f}_{(v,w)} - \sum_{w \in S_2 \setminus S_1} \hat{f}_{(v,w)}) \\ &= \Gamma(S_1 \setminus S_2|E') - \Gamma(S_2 \setminus S_1|E'). \end{aligned} \tag{18}$$

□

Lemma 2.4. For a graph $\mathcal{G}(V, E)$,

$$\Gamma(S_1 \cup S_2|E') + \Gamma(S_1 \cap S_2|E') = \Gamma(S_1|E') + \Gamma(S_2|E'), \quad \forall S_1, S_2 \subseteq V, E' \subseteq E.$$

Proof. It can be easily shown by the fact that $\sum_{v \in (S_1 \cup S_2)} + \sum_{v \in (S_1 \cap S_2)} = \sum_{v \in S_1} + \sum_{v \in S_2}$. □

Let

$$F = \{(r^+, r^-, p, f) \in \mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|V|} \times \mathbb{R}^{|V|} \times \mathbb{R}^{|E|} : (19) - (21)\},$$

where

$$p_v + \delta_v = \sum_{e=(v,\cdot) \in E} f_e - \sum_{e=(\cdot,v) \in E} f_e, \quad v \in V \tag{19}$$

$$-r_v^- \leq p_v \leq r_v^+, \quad v \in V \tag{20}$$

$$-T_e^- \leq f_e \leq T_e^+, \quad e \in E. \tag{21}$$

Let

$$F_p = \{(r^+, r^-, p) \in \mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|V|} \times \mathbb{R}^{|V|} : (22) - (23)\},$$

where

$$-I(S|E) \leq \sum_{v \in S} (p_v + \delta_v) \leq O(S|E), \quad S \in \mathcal{W}(\mathcal{G}) \tag{22}$$

$$-r_v^- \leq p_v \leq r_v^+, \quad v \in V. \tag{23}$$

Let

$$F_r = \{(r^+, r^-) \in \mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|V|} : (24) - (25)\},$$

where

$$\sum_{v \in S} r_v^- \geq \sum_{v \in S} \delta_v - O(S|E), \quad S \in \mathcal{W}(\mathcal{G}) \tag{24}$$

$$\sum_{v \in S} r_v^+ \geq -\sum_{v \in S} \delta_v - I(S|E), \quad S \in \mathcal{W}(\mathcal{G}). \tag{25}$$

Theorem 2.5. $Proj_{(r^+, r^-)}(F) = F_r$.

Proof. The proof will be based on two steps. First, in Claim 2.5.2, we will show that the projection of F onto the space of (r^+, r^-, p) is F_p. Second, in Claim 2.5.1, we will show that the projection of F_p onto the space of (r^+, r^-) is F_r. Claim 2.5.2 and Claim 2.5.1 together imply that $Proj_{(r^+, r^-)}(F) = F_r$. □

Algorithm 1: Finding feasible $\hat{p}$ to F_p from $(\hat{r}^+, \hat{r}^-) \in F_r$

Input: $\mathcal{G} = (V, E)$, $(\hat{r}^+, \hat{r}^-) \in F_r$
Output: $\hat{p}$
Start with an empty set $R \leftarrow \emptyset$;
while $R \neq V$ **do**

1. Choose $v \in V \setminus R$ such that $R \cup v \subseteq \mathcal{W}(\mathcal{G})$;
2. Fix $\hat{p}_v$ satisfying (26) - (28);

$$-\hat{r}_v^- \leq \hat{p}_v \leq \hat{r}_v^+ \tag{26}$$

$$\hat{p}_v \geq -\sum_{w \in R \cap S} \hat{p}_w - \sum_{w \in S \setminus \{R \cup v\}} \hat{r}_w^+ - \sum_{w \in S} \delta_w - I(S|E), \quad S \in \mathcal{W}(\mathcal{G}) : v \in S \tag{27}$$

$$\hat{p}_v \leq -\sum_{w \in R \cap S} \hat{p}_w + \sum_{w \in S \setminus \{R \cup v\}} \hat{r}_w^- - \sum_{w \in S} \delta_w + O(S|E), \quad S \in \mathcal{W}(\mathcal{G}) : v \in S \tag{28}$$

3. $R \leftarrow R \cup v$;

end

Claim 2.5.1. $Proj_{(r^+, r^-)}(F_p) = F_r$.

Proof. First, we show that $Proj_{(r^+, r^-)}(F_p) \subseteq F_r$. From (23),

$$-\sum_{v \in S} r_v^- \leq \sum_{v \in S} p_v \leq \sum_{v \in S} r_v^+. \tag{29}$$

Now it is easy to see that (29) and (22) implies (24) and (25).

Second, we show that $F_r \subseteq Proj_{(r^+, r^-)}(F_p)$. It suffices to show that for all $(\hat{r}^+, \hat{r}^-) \in F_r$, there exists $\hat{p}$ such that $(\hat{r}^+, \hat{r}^-, \hat{p}) \in F_p$. We show that we can find such $\hat{p}$ from Algorithm 1 and it always exists. If it exists, it is easy to show that $\hat{p}$ satisfies (23) from (26). Also, observe that $\hat{p}$ satisfies (22) because for all $S \in \mathcal{W}(\mathcal{G})$, on the course of the while statement, there exists v, R such that $S \not\subseteq R, S \subseteq R \cup v$ then (27) and (28) for S with such v, R become (22).

Now, we show the existence of such $\hat{p}$ in Algorithm 1. We use mathematical induction. Denote R_i and v_i as the node sets and the nodes we get from the Algorithm 1 as it iterates under the while statement. For the first step we consider the case where $R_1 = \emptyset$. The lower bound of (27) $\leq$ the upper bound of (26) is implied by (25) and the upper bound of (28) $\geq$ the lower bound of (26) is implied by (24). For showing why the lower bound of (27) $\leq$ the upper bound of (28), pick $S_1, S_2 \in \{S \in \mathcal{W}(\mathcal{G}) : v_1 \in S\}$. From (24) for $S_2 \setminus S_1$ and (25) for $S_1 \setminus S_2$ [1] using Lemma 2.1,

$$\begin{aligned} \sum_{w \in S_1 \setminus S_2} r_w^+ + \sum_{w \in S_2 \setminus S_1} r_w^- &\geq -\sum_{w \in S_1 \setminus S_2} \delta_w + \sum_{w \in S_2 \setminus S_1} \delta_w - I(S_1 \setminus S_2|E) - O(S_2 \setminus S_1|E) \\ &\geq -\sum_{w \in S_1 \setminus S_2} \delta_w + \sum_{w \in S_2 \setminus S_1} \delta_w - I(S_1|E) - O(S_2|E). \end{aligned} \tag{30}$$

Since $\sum_{w \in (S_1 \cap S_2) \setminus v_1} (r_w^+ + r_w^-) \geq 0$, (30) implies

$$\sum_{w \in S_1 \setminus v_1} r_w^+ + \sum_{w \in S_2 \setminus v_1} r_w^- \geq -\sum_{w \in S_1} \delta_w + \sum_{w \in S_2} \delta_w - I(S_1|E) - O(S_2|E), \tag{31}$$

which is equivalent to the lower bound of (27) for $S_1 \leq$ the upper bound of (28) for S_2. Thus, $\hat{p}_{v_1}$ satisfying (26) - (28) exists for the case where $R_1 = \emptyset$.

[1] It is possible that $S_1 \setminus S_2 \notin \mathcal{W}(\mathcal{G})$ or $S_2 \setminus S_1 \notin \mathcal{W}(\mathcal{G})$, but in this case there exists disjoint $S_A, S_B \in \mathcal{W}(\mathcal{G})$ such that $S_A \cup S_B = S_1 \setminus S_2$ or $S_A \cup S_B = S_2 \setminus S_1$, and we can get the same results as (30) by summing up (24) or (25) for S_A and that for S_B.

For the next step of mathematical induction, assume that for $i \geq 1$, there exists $\hat{p}_{v_k}$ for $1 \leq k \leq i$ satisfying (26) - (28). For $R_{i+1} = R_i \cup v_i$ and $v_{i+1} \in V \setminus R_{i+1}$, our goal is to show that all the possible combinations of the upper bounds and the lower bounds from (26) - (28) can be implied by other inequalities so that we can show that $\hat{p}_{v_{i+1}}$ exists. First, we show it for the combinations of upper bounds and lower bounds between (26) and (27) - (28). Here, we show one out of the two cases: the lower bound of (27) $\leq$ the upper bound of (26). The other case can be shown in a similar fashion. The set $\mathcal{W}(\mathcal{G})$ can be divided into two cases : i) $R_{i+1} \cap S = \emptyset$ and ii) $R_{i+1} \cap S \neq \emptyset$. For the case i), $\sum_{w \in R_{i+1} \cap S} \hat{p}_w = 0$ and $\sum_{w \in S \setminus \{R_{i+1} \cup v_{i+1}\}} \hat{r}^+_w = \sum_{w \in S \setminus v_{i+1}} \hat{r}^+_w$, so (25) implies the lower bound of (27) $\leq$ the upper bound of (26). For the case ii), from the set $\{v : v \in R_{i+1} \cap S\}$, pick the node with the largest index l. Observe that $\sum_{w \in R_{i+1} \cap S} \hat{p}_w = \sum_{w \in R_l \cap S} \hat{p}_w + \hat{p}_{v_l}$ and $\sum_{w \in S \setminus R_{i+1}} \hat{r}^+_w = \sum_{w \in S \setminus \{R_l \cup v_l\}} \hat{r}^+_w$. This can be proved by contradiction. Assume that it is not true. Then $\exists v_m$ such that $m \neq l, v_m \in R_{i+1}, v_m \notin R_l$, and $v_m \in S$. This contradicts the fact that l is the largest index. Thus, (27) with R_l and v_l implies the lower bound of (27) $\leq$ the upper bound of (26).

For showing why the lower bound of (27) $\leq$ the upper bound of (28), pick $S_1, S_2 \in \{S \in \mathcal{W}(\mathcal{G}) : v \in S\}$. We have four different cases to show : i) $R_{i+1} \cap S_1 = \emptyset$, $R_{i+1} \cap S_2 = \emptyset$, ii) $R_{i+1} \cap S_1 \neq \emptyset$, $R_{i+1} \cap S_2 = \emptyset$, iii) $R_{i+1} \cap S_1 = \emptyset$, $R_{i+1} \cap S_2 \neq \emptyset$, iv) $R_{i+1} \cap S_1 \neq \emptyset$, $R_{i+1} \cap S_2 \neq \emptyset$. Since it is similar to prove another cases, here we show for the case ii) where $R_{i+1} \cap S_1 \neq \emptyset$, $R_{i+1} \cap S_2 = \emptyset$. From the set $\{v : v \in R_{i+1} \cap (S_1 \setminus S_2)\}$, pick the node with the largest index l. Similar to what we have shown above, observe that $\sum_{w \in R_{i+1} \cap (S_1 \setminus S_2)} \hat{p}_w = \sum_{w \in R_l \cap (S_1 \setminus S_2)} \hat{p}_w + \hat{p}_{v_l}$ and $\sum_{w \in (S_1 \setminus S_2) \setminus R_{i+1}} \hat{r}^+_w = \sum_{w \in (S_1 \setminus S_2) \setminus \{R_l \cup v_l\}} \hat{r}^+_w$. From (27) for $S_1 \setminus S_2$ with R_l, v_l and (24) for $S_2 \setminus S_1$ using Lemma 2.1, following the similar process in (30) and (31) we get the inequality,

$$\sum_{w \in R_{i+1} \cap S_1} \hat{p}_w + \sum_{w \in S_1 \setminus R_{i+1}} r^+_w + \sum_{w \in S_2} r^-_w \geq - \sum_{w \in S_1} \delta_w + \sum_{w \in S_2} \delta_w - I(S_1|E) - O(S_2|E), \tag{32}$$

which is equivalent to the lower bound of (27) for $S_1 \leq$ the upper bound of (28) for S_2.

Thus, $\hat{p}_{v_{i+1}}$ satisfying (26) - (28) exists and it proves the existence of $\hat{p}$.

□

Claim 2.5.2. $Proj_{(r^+, r^-, p)}(F) = F_p$.

Proof. First, we show that $Proj_{(r^+, r^-, p)}(F) \subseteq F_p$. Notice that (20) and (23) are identical. So, it suffices to show that (19) and (21) implies (22). From (19),

$$\sum_{v \in S} (p_v + \delta_v) = \sum_{v \in S, w \in S^c} f_{(v,w)} - \sum_{v \in S, w \in S^c} f_{(w,v)}, \quad S \in \mathcal{W}(\mathcal{G}). \tag{38}$$

Now, it is easy to see that (38) and (21) implies (22).

Second, we show that $F_p \subseteq Proj_{(r^+, r^-, p)}(F)$. It suffices to show that for all $(\hat{r}^+, \hat{r}^-, \hat{p}) \in F_p$, there exists $\hat{f}$ such that $(\hat{r}^+, \hat{r}^-, \hat{p}, \hat{f}) \in F$. We show that we can find such $\hat{f}$ from Algorithm 2 and it always exists. If it exists, it is easy to show that $\hat{f}$ satisfies (21) from (33). Also, observe that $\hat{f}$ satisfies (19) from (34) - (37). For all $v \in V$, let $E(v) = \{e \in E : e = (v, \cdot) \cup e = (\cdot, v)\}$. On the course of Algorithm 2, when we pick (v, w) such that $E(v) \subset Q \cup (v, w)$, with such Q and $S = \{v\}$, (34) and (35) become (19). Likewise, when we pick (w, v) such that $E(v) \subset Q \cup (w, v)$, with such Q and $S = \{v\}$, (36) and (37) become (19).

Now, we show the existence of such $\hat{f}$ in Algorithm 2. We use mathematical induction. Denote Q_i and (v_i, w_i) as the edge sets and the edges we get from the Algorithm 2 as it iterates under the while statement. For the first step we consider the case where $Q_1 = \emptyset$. Then, $\Gamma(S|Q_1) = 0$ for all $S \in \mathcal{W}(\mathcal{G})$. We want to show that (22) implies all the possible combinations of upper bounds and lower bounds among (33) - (37). First we show for the combinations between (33) and (34) - (37).

Algorithm 2: Finding feasible $\hat{f}$ to F from $(\hat{r}^+, \hat{r}^-, \hat{p}) \in F_p$

Input: $\mathcal{G} = (V, E)$, $(\hat{r}^+, \hat{r}^-, \hat{p}) \in F_p$

Output: $\hat{f}$

Start with an empty set $Q \leftarrow \emptyset$;

while $Q \neq E$ **do**

1. Choose $(v, w) \in E \setminus Q$;
2. Fix $\hat{f}_{(v,w)}$ satisfying (33) - (37);

$$-T^-_{(v,w)} \leq \hat{f}_{(v,w)} \leq T^+_{(v,w)} \tag{33}$$

For all $S \in \mathcal{W}(\mathcal{G}) : v \in S, w \notin S$

$$\hat{f}_{(v,w)} \geq \sum_{u \in S} (\hat{p}_u + \delta_u) - \Gamma(S|Q) - O(S|E) + O(S|Q \cup (v, w)) \tag{34}$$

$$\hat{f}_{(v,w)} \leq \sum_{u \in S} (\hat{p}_u + \delta_u) - \Gamma(S|Q) + I(S|E) - I(S|Q \cup (v, w)) \tag{35}$$

For all $S \in \mathcal{W}(\mathcal{G}) : v \notin S, w \in S$

$$\hat{f}_{(v,w)} \geq -\sum_{u \in S} (\hat{p}_u + \delta_u) + \Gamma(S|Q) - I(S|E) + I(S|Q \cup (v, w)) \tag{36}$$

$$\hat{f}_{(v,w)} \leq -\sum_{u \in S} (\hat{p}_u + \delta_u) + \Gamma(S|Q) + O(S|E) - O(S|Q \cup (v, w)) \tag{37}$$

3. $Q \leftarrow Q \cup (v, w)$;

end

As an example, in (34), $O(S|Q_1 \cup (v_1, w_1)) = T^+_{(v_1,w_1)}$. So, (22) implies the upper bound of (33) $\geq$ the lower bound of (34). In a similar way, we can show that (22) implies all the possible combinations of upper bounds and lower bounds between (33) and (35) - (37). Next, we still need to show why the lower bound of (34) $\leq$ the upper bound of (35), and for the case of (36) and (37). Since the pattern is similar, we show that of (34) and (35) as an example. Pick $S_1, S_2 \in \{S \in \mathcal{W}(\mathcal{G}) : v_1 \in S, w_1 \notin S\}$. Notice the equation (39) holds because $v_1 \in S_1 \cap S_2$, which implies $v_1 \notin S_1 \setminus S_2$ and $v_1 \notin S_2 \setminus S_1$.

$$O(S_1 \setminus S_2|E) + I(S_2 \setminus S_1|E) = O(S_1 \setminus S_2|E \setminus (v_1, w_1)) + I(S_2 \setminus S_1|E \setminus (v_1, w_1)) \tag{39}$$

From (22) for $S_1 \setminus S_2$ and $S_2 \setminus S_1$ using the equation (39) and Lemma 2.1,

$$\begin{aligned} 0 &\leq O(S_1 \setminus S_2|E) + I(S_2 \setminus S_1|E) - \sum_{u \in S_1 \setminus S_2} (\hat{p}_u + \delta_u) + \sum_{u \in S_2 \setminus S_1} (\hat{p}_u + \delta_u) \\ &= O(S_1 \setminus S_2|E \setminus (v_1, w_1)) + I(S_2 \setminus S_1|E \setminus (v_1, w_1)) - \sum_{u \in S_1} (\hat{p}_u + \delta_u) + \sum_{u \in S_2} (\hat{p}_u + \delta_u) \\ &\leq O(S_1|E \setminus (v_1, w_1)) + I(S_2|E \setminus (v_1, w_1)) - \sum_{u \in S_1} (\hat{p}_u + \delta_u) + \sum_{u \in S_2} (\hat{p}_u + \delta_u), \end{aligned} \tag{40}$$

which is equivalent to the lower bound of (34) for $S_1 \leq$ the upper bound of (35) for S_2. Now, the remaining combinations are the lower bound of (34) $\leq$ the upper bound of (37) and the case of (35) and (36). Here, we show that of (34) and (37). Pick $S_1 \in \{S \in \mathcal{W}(\mathcal{G}) : v_1 \in S, w_1 \notin S\}$ and $S_2 \in \{S \in \mathcal{W}(\mathcal{G}) : v_1 \notin S, w_1 \in S\}$. Notice that $(S_1 \cup S_2) \in \mathcal{W}(\mathcal{G})$ because (v_1, w_1) connects S_1 and S_2, and $(S_1 \cap S_2) \in \mathcal{W}(\mathcal{G})$. From (22) for $(S_1 \cup S_2)$ and $(S_1 \cap S_2)$ using Lemma 2.2,

$$\begin{aligned} \sum_{u \in S_1} (\hat{p}_u + \delta_u) + \sum_{u \in S_2} (\hat{p}_u + \delta_u) &\leq O(S_1 \cup S_2|E) + O(S_1 \cap S_2|E) \\ &\leq O(S_1|E) + O(S_2|E) - T^+_{(v_1,w_1)} - T^-_{(v_1,w_1)}, \end{aligned} \tag{41}$$

which is equivalent to the lower bound of (34) for $S_1 \leq$ the upper bound of (37) for S_2.

For the next step of mathematical induction, assume that for $i \geq 1$, there exists $\hat{f}_{(v_k,w_k)}$ for $1 \leq k \leq i$ satisfying (33) - (37). For $Q_{i+1} = Q_i \cup (v_i, w_i)$ and $(v_{i+1}, w_{i+1}) \in E \setminus Q_{i+1}$, our goal is to show that all the possible combinations of the upper bounds and the lower bounds from (33) - (37) can be implied by other inequalities so that we can show that $\hat{f}_{(v_{i+1},w_{i+1})}$ exists. First we show for the combinations of upper bounds and lower bounds between (33) and (34) - (37). Here, we show one out of the four cases: the upper bound of (33) $\geq$ the lower bound of (34). The other three cases can be shown in a similar fashion. The set $\{S \in \mathcal{W}(\mathcal{G}) : v_{i+1} \in S, w_{i+1} \notin S\}$ can be divided into two cases: i) $\Gamma(S|Q_{i+1}) = 0$, and ii) $\Gamma(S|Q_{i+1}) \neq 0$.

For the case $\Gamma(S|Q_{i+1}) = 0$, $O(S|Q_{i+1} \cup (v_{i+1}, w_{i+1})) = T^+_{(v_{i+1},w_{i+1})}$. So, (22) implies the upper bound of (33) $\geq$ the lower bound of (34). For the case $\Gamma(S|Q_{i+1}) \neq 0$, it is equivalent to say that the set $\{(v_k, w_k) : k \leq i$ such that $v_k \in S \cap V(Q_{i+1}), w_k \in V(Q_{i+1}) \setminus S$ or $w_k \in S \cap V(Q_{i+1}), v_k \in V(Q_{i+1}) \setminus S\}$ is nonempty. Pick the edge with the largest index l from this set. If $v_l \in S \cap V(Q_{i+1}), w_l \in V(Q_{i+1}) \setminus S$, then $\Gamma(S|Q_l) + \hat{f}_{(v_l,w_l)} = \Gamma(S|Q_{i+1})$ and $O(S|Q_{i+1}) = O(S|Q_l \cup (v_l, w_l))$. This can be proved by contradiction. Assume that it is not true. Then $\exists(v_m, w_m)$ such that $m \neq l, (v_m, w_m) \in Q_{i+1}, (v_m, w_m) \notin Q_l$ and $v_m \in S, w_m \in V(Q_{i+1}) \setminus S$. This contradicts the fact that l is the largest index. Observe that $O(S|Q_{i+1} \cup (v_{i+1}, w_{i+1})) = O(S|Q_{i+1}) + T^+_{(v_{i+1},w_{i+1})} = O(S|Q_l \cup (v_l, w_l)) + T^+_{(v_{i+1},w_{i+1})}$. Thus, (34) with Q_l and (v_l, w_l) implies the upper bound of (33) $\geq$ the lower bound of (34). If $w_l \in S \cap V(Q_{i+1}), v_l \in V(Q_{i+1}) \setminus S$, then $\Gamma(S|Q_l) - \hat{f}_{(v_l,w_l)} = \Gamma(S|Q_{i+1})$ and $O(S|Q_{i+1}) = O(S|Q_l \cup (v_l, w_l))$. Similarly, (37) with Q_l and (v_l, w_l) implies the upper bound of (33) $\geq$ the lower bound of (34).

Next, we still need to show that the lower bound of (34) $\leq$ the upper bound of (35) and for the case of (36) and (37). Since the way is similar, we show the case of (34) and (35) as an example. Pick $S_1, S_2 \in \{S \in \mathcal{W}(\mathcal{G}) : v_{i+1} \in S, w_{i+1} \notin S\}$. There are four cases to show: i) $\Gamma(S_1 \setminus S_2|Q_{i+1}) = 0, \Gamma(S_2 \setminus S_1|Q_{i+1}) = 0$, ii) $\Gamma(S_1 \setminus S_2|Q_{i+1}) \neq 0, \Gamma(S_2 \setminus S_1|Q_{i+1}) = 0$, iii) $\Gamma(S_1 \setminus S_2|Q_{i+1}) = 0, \Gamma(S_2 \setminus S_1|Q_{i+1}) \neq 0$, iv) $\Gamma(S_1 \setminus S_2|Q_{i+1}) \neq 0, \Gamma(S_2 \setminus S_1|Q_{i+1}) \neq 0$. Here, we show for the case ii) as an example since another cases are similar to that. As mentioned above, $\Gamma(S_1 \setminus S_2|Q_{i+1}) \neq 0$ is equivalent to say that the set $\{(v_k, w_k) : k \leq i$ such that $v_k \in (S_1 \setminus S_2) \cap V(Q_{i+1}), w_k \in V(Q_{i+1}) \setminus (S_1 \setminus S_2)$ or $w_k \in (S_1 \setminus S_2) \cap V(Q_{i+1}), v_k \in V(Q_{i+1}) \setminus (S_1 \setminus S_2)\}$ is nonempty. From this set, pick the edge with the largest index l. Without loss of generality, we assume that $v_l \in (S_1 \setminus S_2) \cap V(Q_{i+1}), w_l \in V(Q_{i+1}) \setminus (S_1 \setminus S_2)$. Then, $\Gamma(S_1 \setminus S_2|Q_l) + \hat{f}_{(v_l,w_l)} = \Gamma(S_1 \setminus S_2|Q_{i+1})$ and $O(S_1 \setminus S_2|Q_{i+1}) = O(S_1 \setminus S_2|Q_l \cup (v_l, w_l))$. Additionally, this case is further divided into two sub-cases : ii-i) $\Gamma(S_2|Q_{i+1}) = 0$, and ii-ii) $\Gamma(S_2|Q_{i+1}) \neq 0$.

For the case ii-i), because of the Lemma 2.3, $\Gamma(S_1 \setminus S_2|Q_{i+1}) = \Gamma(S_1|Q_{i+1})$. From (34) for $S_1 \setminus S_2$ with Q_l, (v_l, w_l) and (22) for $S_2 \setminus S_1$,

$$0 \leq \Gamma(S_1 \setminus S_2|Q_{i+1}) + O(S_1 \setminus S_2|E \setminus Q_{i+1}) + I(S_2 \setminus S_1|E) - \sum_{u \in S_1 \setminus S_2} (\hat{p}_u + \delta_u) + \sum_{u \in S_2 \setminus S_1} (\hat{p}_u + \delta_u). \quad (42)$$

Notice that the equations (43) hold because $v_{i+1} \notin (S_1 \setminus S_2)$,$v_{i+1} \notin (S_2 \setminus S_1)$, and $\Gamma(S_2 \setminus S_1|Q_{i+1}) = 0$.

$$\begin{aligned} O(S_1 \setminus S_2|E \setminus Q_{i+1}) &= O(S_1 \setminus S_2|E \setminus (Q_{i+1} \cup (v_{i+1}, w_{i+1}))) \\ I(S_2 \setminus S_1|E) &= I(S_2 \setminus S_1|E \setminus (Q_{i+1} \cup (v_{i+1}, w_{i+1}))) \end{aligned} \quad (43)$$

Using the equations (43), Lemma 2.1 and the fact that $\Gamma(S_1 \setminus S_2|Q_{i+1}) = \Gamma(S_1|Q_{i+1})$, (42) implies the lower bound of (34) for $S_1 \leq$ the upper bound of (35) for S_2. For the case ii-ii), because of the Lemma 2.3, $\Gamma(S_1 \setminus S_2|Q_{i+1}) = \Gamma(S_1|Q_{i+1}) - \Gamma(S_2|Q_{i+1})$. Similar to the case ii-i), we can show that (34) for $S_1 \setminus S_2$ with Q_l, (v_l, w_l) and (22) for $S_2 \setminus S_1$ imply the lower bound of (34) for $S_1 \leq$ the upper bound of (35) for S_2.

Lastly, we need to show that the lower bound of (34) $\leq$ the upper bound of (37) and the case of (35) and (36). Since the pattern is similar, we show the case of (34) and (37) as an example. Pick $S_1 \in \{S \in \mathcal{W}(\mathcal{G}) : v_{i+1} \in S, w_{i+1} \notin S\}$ and $S_2 \in \{S \in \mathcal{W}(\mathcal{G}) : v_{i+1} \notin S, w_{i+1} \in S\}$. There are four cases to show: i) $\Gamma(S_1\cup S_2|Q_{i+1})=0, \Gamma(S_1\cap S_2|Q_{i+1})=0$, ii) $\Gamma(S_1\cup S_2|Q_{i+1})\neq 0, \Gamma(S_1\cap S_2|Q_{i+1})=0$, iii) $\Gamma(S_1\cup S_2|Q_{i+1})=0, \Gamma(S_1\cap S_2|Q_{i+1})\neq 0$, iv) $\Gamma(S_1\cup S_2|Q_{i+1})\neq 0, \Gamma(S_1\cap S_2|Q_{i+1})\neq 0$. Here, we show for the case ii) as an example since another cases are similar to that. As mentioned above, $\Gamma(S_1\cup S_2|Q_{i+1})\neq 0$ is equivalent to say that the set $\{(v_k,w_k): k\leq i$ such that $v_k\in (S_1\cup S_2)\cap V(Q_{i+1}), w_k\in V(Q_{i+1})\backslash(S_1\cup S_2)$ or $w_k\in(S_1\cup S_2)\cap V(Q_{i+1}), v_k\in V(Q_{i+1})\backslash(S_1\cup S_2)\}$ is nonempty. From this set, pick the edge with the largest index l. Without loss of generality, we assume that $v_l\in(S_1\cup S_2)\cap V(Q_{i+1}), w_l\in V(Q_{i+1})\backslash(S_1\cup S_2)$. Then, $\Gamma(S_1\cup S_2|Q_l)+\hat{f}_{(v_l,w_l)}=\Gamma(S_1\cup S_2|Q_{i+1})$ and $O(S_1\cup S_2|Q_{i+1})=O(S_1\cup S_2|Q_l\cup(v_l,w_l))$. From (34) for $(S_1\cup S_2)$ with $Q_l,(v_l,w_l)$ and (22) for $(S_1\cap S_2)$,

$$\sum_{u\in S_1}(\hat{p}_u+\delta_u)+\sum_{u\in S_2}(\hat{p}_u+\delta_u)\leq\Gamma(S_1\cup S_2|Q_{i+1})+O(S_1\cup S_2|E\setminus Q_{i+1})+O(S_1\cap S_2|E). \tag{44}$$

Notice that the Lemma 2.4 and the fact that $\Gamma(S_1\cap S_2|Q_{i+1})=0$ imply $\Gamma(S_1\cup S_2|Q_{i+1})=\Gamma(S_1|Q_{i+1})+\Gamma(S_2|Q_{i+1})$. Since $v_{i+1},w_{i+1}\in(S_1\cup S_2)$, $v_{i+1},w_{i+1}\notin(S_1\cap S_2)$, and $\Gamma(S_1\cap S_2|Q_{i+1})=0$,

$$\begin{aligned}O(S_1\cup S_2|E\setminus Q_{i+1})&=O(S_1\cup S_2|E\setminus(Q_{i+1}\cup(v_{i+1},w_{i+1})))\\ O(S_1\cap S_2|E)&=O(S_1\cap S_2|E\setminus(Q_{i+1}\cup(v_{i+1},w_{i+1}))).\end{aligned} \tag{45}$$

Using the equations (45), Lemma 2.2, and the fact that $\Gamma(S_1\cup S_2|Q_{i+1})=\Gamma(S_1|Q_{i+1})+\Gamma(S_2|Q_{i+1})$, (44) implies the lower bound of (34) for $S_1\leq$ the upper bound of (37) for S_2.

Thus, $\hat{f}_{(v_{i+1},w_{i+1})}$ satisfying (33) - (37) exists and it proves the existence of $\hat{f}$. □

Theorem 2.6. *F_r is a minimal representation on the space of (r^+,r^-).*

Proof. Since the proof for the set of inequalities (24), is similar to the case for (25), we show here the case for (24). In order to show that (24) is a minimal representation on the space of r^-, in order to show the contradiction in the end, first let us assume that there exists a set $S'\in\mathcal{W}(\mathcal{G})$ such that there exist mutually different sets $S_1',\ldots,S_n'\in\mathcal{W}(\mathcal{G})$ by which the inequality constructed of the form (24) dominates the inequality for the set S'. In a mathematical expression, it means that there exist coefficients $\alpha_1,\ldots,\alpha_n\geq 0$ that satisfies the following conditions (46) and (47).

$$\alpha_1\sum_{v\in S_1'}r_v^-+\cdots+\alpha_n\sum_{v\in S_n'}r_v^-\leq\sum_{v\in S'}r_v^- \tag{46}$$

$$\alpha_1\sum_{v\in S_1'}\delta_v+\cdots+\alpha_n\sum_{v\in S_n'}\delta_v-\alpha_1 O(S_1'|E)-\cdots-\alpha_n O(S_n'|E)\geq\sum_{v\in S'}\delta_v-O(S'|E) \tag{47}$$

In order to satisfy the inequalities (46) and (47) for all possible values of r_v^- and δ_v, $\sum_{i:v\in S_i'}\alpha_i=1$ for all $v\in S'$ and $\sum_{i:v\in S_i'}\alpha_i=0$ for all $v\in V\setminus S'$. This implies that for all $i\in\{1,\ldots,n\}$, $S_i'\subseteq S'$ and $\bigcup_{i=1}^n S_i'=S'$. Notice that the left-hand side and the right-hand side for of (46) are equal, and (47) becomes

$$O(S'|E)\geq\alpha_1 O(S_1'|E)+\cdots+\alpha_n O(S_n'|E). \tag{48}$$

Since the right-hand side of (48)

$$\begin{aligned}\alpha_1 O(S_1'|E)+\cdots+\alpha_n O(S_n'|E)&=\sum_{i:v\in S_i'}\alpha_i\cdot\Big(\sum_{v\in S',w\in(S')^c}T^+_{(v,w)}+\sum_{v\in S',w\in(S')^c}T^-_{(w,v)}\Big)+\tilde{O}\\ &=O(S'|E)+\tilde{O},\end{aligned}$$

where $\tilde{O}=\sum_{i=1}^n\sum_{j=1}^n\alpha_i\Big(\sum_{v\in S_i',w\in(S_i')^c\cap S_j}T^+_{(v,w)}+\sum_{v\in S_i',w\in(S_i')^c\cap S_j}T^-_{(w,v)}\Big)>0$, it contradicts the initial assumption. □